JOURNAL OF COMBINATORIAL THEORY, Series A 24, 118–121 (1978)

Note

Ordered Sums of Group Elements

EDWARD T. WHITE

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802

Communicated by the Managing Editors

Received December 23, 1975

The following theorem is proved. If $a_1, \ldots, a_k$ are distinct elements of a group, written additively, though not necessarily Abelian, and the sums $a_{i_1} + \cdots + a_{i_m}$, $1 \leqslant i_1 < \cdots < i_m \leqslant k$ do not represent 0, then they represent at least $2k - 1$ distinct elements, and this bound $2k - 1$ is attained only when $k \leqslant 3$ or when the elements $a_1, \ldots, a_k$ generate a dihedral group.

1

Let G be a group written additively, though not necessarily Abelian. For a sequence $S = (a_1, \ldots, a_k)$ of elements of G, define $\sum(S) = \sum(a_1, \ldots, a_k)$ to be the set of all sums $a_{i_1} + a_{i_2} + \cdots + a_{i_m}$ with $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant k$ and $m > 0$. If B is a finite subset of G, let $|B|$ denote the number of elements of B. Eggleton and Erdös [1] proved that if G is Abelian, and S is a sequence of k distinct elements of G with $0 \notin \sum(S)$, then $|\sum(S)| \geqslant 2k - 1$ in general, and $|\sum(S)| \geqslant 2k$ for $k \geqslant 4$. In this paper, we prove the following:

THEOREM 1.1. *If $S = (a_1, \ldots, a_k)$ is a sequence of k distinct elements of a group G and $0 \notin \sum(S)$, then*

(1) $|\sum(S)| \geqslant 2k - 1$.

(2) *If $k \geqslant 4$, then either $|\sum(S)| \geqslant 2k$ or $a_1, \ldots, a_k$ generate a dihedral group, possibly infinite.*

The lower bound $2k - 1$ in Theorem 1.1 can actually be attained in a dihedral group as shown by the following example. Consider the dihedral group generated by a and b, where $2a = 0$, $b + a = a - b$, and b has order at least k, possibly infinite. Let $S = (a, a + b, a + 2b, \ldots, a + (k - 1)b)$. Then $0 \notin \sum(S)$ and $|\sum(S)| = 2k - 1$.

0097-3165/78/0241-0118$02.00/0

Szemerédi [4] showed that there is an absolute constant c such that $|\sum(S)| \geqslant ck^2$ for any sequence S of k distinct elements of an Abelian group with $0 \notin \sum(S)$. Olson [3] proved that if T is a set of k distinct elements of an arbitrary group, then the elements of T *can be arranged* into a sequence $S = (a_1, a_2, ..., a_k)$ such that either $0 \in \sum(S)$ or $|\sum(S)| \geqslant \frac{1}{9}k^2$. The above example shows that Szemerédi's theorem does not carry over to non-Abelian groups, i.e., Olson's result does not extend to a given arrangement of the a_i.

2

The subgroup of G generated by $\{g_1, ..., g_n\} \subset G$ will be denoted by $\langle g_1, ..., g_n \rangle$, and the order of $g \in G$ will be denoted by $O(g)$. We require the following two theorems:

THEOREM (Eggleton and Erdös [1]). *Let S be a sequence of k distinct elements of an Abelian group, such that $0 \notin \sum(S)$. Then $|\sum(S)| \geqslant 2k - 1$, and if $k \geqslant 4$, then $|\sum(S)| \geqslant 2k$.*

THEOREM (*Kemperman and Wehn* [2]). *Let A and B be finite nonempty subsets of a group G. If some element $c \in A + B$ has exactly one representation as a sum $c = a + b$ with $a \in A$, $b \in B$, then $|A + B| \geqslant |A| + |B| - 1$.*

The following proposition is an immediate consequence of the Kemperman–Wehn theorem.

PROPOSITION 2.1. *Let $S = (a_1, ..., a_k)$ be a sequence of distinct elements of a group G with $0 \notin \sum(S)$. Then for any positive integer $t < k$, $|\sum(S)| \geqslant |\sum(a_1, ..., a_t)| + |\sum(a_{t+1}, ..., a_k)|$.*

Proof. 0 has only one representation in $[\{0\} \cup \sum(a_1, ..., a_t)] + [\{0\} \cup \sum(a_{t+1}, ..., a_k)]$.

PROPOSITION 2.2. *Let $S = (a_1, ..., a_k)$ be a sequence of elements of a group G with $0 \notin \sum(S)$, and let $H = \{g \in G: \sum(S) + g = \sum(S)\}$. Then $|\sum(S)| \geqslant k\,|H|$.*

Proof. Clearly, H is a finite subgroup of G. We have $\sum(S) + H = \sum(S)$. Since $0 \notin \sum(S)$, $\sum(S)$ is a union of nonzero left cosets of H. For $1 \leqslant u \leqslant k$, let $c_u = \sum_{i=1}^{u} a_i$. If $u < v$, then $-c_u + c_v \in \sum(S)$, hence $-c_u + c_v \notin H$. Thus, $c_1, c_2, ..., c_k$ lie in different left cosets of H, hence $|\sum(S)| \geqslant k\,|H|$.

PROPOSITION 2.3. *Let $S = (a_1, ..., a_k)$ be a sequence of $k \geqslant 2$ distinct elements of a group G with $0 \notin \sum(S)$, and $|\sum(S)| < 2k$. If $|\sum(a_i, a_{i+1}, ...,$*

$a_{i+m})| \geqslant 2m + 1$ for $1 \leqslant i \leqslant i + m \leqslant k$, with $m \leqslant k - 2$, then $\langle a_1, ..., a_k \rangle = \langle a_1, a_2 \rangle = \langle a_{k-1}, a_k \rangle$.

Proof. Suppose $\langle a_1, ..., a_k \rangle \neq \langle a_1, a_2 \rangle$, so $k \geqslant 3$. Let t be the smallest index such that $a_t \notin \langle a_1, a_2 \rangle$. We have $3 \leqslant t \leqslant k$. Then $\sum (a_1, ..., a_{t-1})$, $\sum (a_1, ..., a_{t-1}) + a_t$, and $\{a_t\}$ are disjoint, hence $|\sum (a_1, ..., a_t)| \geqslant 2 |\sum (a_1, ..., a_{t-1})| + 1 \geqslant 4t - 5$. If $t = k$, this contradicts the hypothesis, for with $k \geqslant 3$ it implies $|\sum (S)| > 2k$. So suppose $t < k$. Then by Proposition 2.1, $|\sum (S)| \geqslant 4t - 5 + |\sum (a_{t+1}, ..., a_k)| \geqslant 2k + 2t - 6 \geqslant 2k$, contrary to hypothesis. This proves $\langle a_1, ..., a_k \rangle = \langle a_1, a_2 \rangle$. A similar argument gives the second equality.

Theorem 1.1 will now follow from the following proposition. Its proof, which involves looking at a number of separate cases, may be found in [5].

PROPOSITION 2.4. *Let $S = (a_1, ..., a_k)$ be a sequence of distinct elements of a group G with $0 \notin \sum (S)$ and $4 \leqslant k \leqslant 7$. Then either $|\sum (S)| \geqslant 2k$ or $\langle a_1, ..., a_k \rangle$ is a dihedral group.*

To prove statement (1) of Theorem 1.1, we proceed by induction on k. The assertion is obvious for $k = 1, 2$. Suppose $k = 3$. Let $A = \sum (a_1, a_2)$. $|A| = 3$, so Proposition 2.2 implies $A + a_3 \neq A$. If $|(A + a_3)\backslash A| \geqslant 2$, we are finished, so now assume $|(A + a_3)\backslash A| = 1$. Now $0 \notin A$ gives $a_3 \notin A + a_3$ hence if $a_3 \notin A$, then $|\sum (S)\backslash A| \geqslant 2$, and the result holds. So now assume $a_3 \in A$; thus $a_3 = a_1 + a_2$. We shall show that this leads to a contradiction. Since $a_3 \notin A + a_3$ and $|(A + a_3)\backslash A| = 1$, it follows that $\{a_1, a_2\} \subseteq A + a_3$. Hence $a_1 = a_2 + a_3$, and $a_2 = a_1 + a_2 + a_3$ since $0 \notin A + a_3$ ensures $2a_3 \neq 0$. Thus $a_2 = 2a_3$ and $a_1 = 3a_3$. But $a_3 = a_1 + a_2 = 5a_3$ shows $4a_3 = 0$, whence $a_1 + a_3 = 0 \in \sum (S)$, contrary to hypothesis.

We may now assume that $k \geqslant 4$, and that the result holds for all sequences of length less than k. We may assume $\langle a_1, ..., a_k \rangle = \langle a_1, a_2 \rangle$, since otherwise, by Proposition 2.3, $|\sum (S)| \geqslant 2k$. By Proposition 2.4, either $\langle a_1, a_2 \rangle$ is a dihedral group or $|\sum (a_1, a_2, a_3, a_4)| \geqslant 8$. In the latter case, the result follows by Proposition 2.1 and induction. Therefore, assume that $a_1, ..., a_k$ are elements of a dihedral group generated by, say, a and b with $O(a) = 2$, $O(b) \geqslant 3$. If $2a_i \neq 0$ for $1 \leqslant i \leqslant k$, then $a_1, ..., a_k \in \langle b \rangle$, and the result follows by the Eggleton–Erdös theorem. Thus we assume $2a_i = 0$ for some i. Suppose $i = k$. Let $B = \sum (a_1, ..., a_{k-1})$. If $B + a_k = B$, then by Proposition 2.2, $|B| \geqslant 2k - 2$. If $b_1, b_2, ..., b_q$ are the distinct elements of B, then $a_k, b_1 + a_k, b_2 + a_k, ..., b_q + a_k$ are distinct elements of $\sum (S)$, so $|\sum (S)| \geqslant q + 1 \geqslant 2k - 1$. Thus we may assume $|(B + a_k)\backslash B| \geqslant 1$. But $O(a_k) = 2$, $0 \notin \sum (S)$ give $a_k \notin B$, $a_k \notin B + a_k$, so $|\sum (S)\backslash B| \geqslant 2$, and we are finished, since $|B| \geqslant 2k - 2$. Therefore, we may assume $i \neq k$, and similarly, $i \neq 1$. In this case, since $\{0\} \cup \sum (S) = [\{0\} \cup \sum (a_1, ..., a_i)] + \sum (a_i, ..., a_k)$ and 0 has only one representation in this sum, induction and the Kemperman–Wehn theorem give $|\sum(a_1,..., a_k)| \geqslant 2k - 1$, and we are finished.

To prove statement (2) of Theorem 1.1 we may, by Proposition 2.4, take $k \geqslant 8$. An argument similar to the proof of Proposition 2.3 shows that if $|\sum(S)| < 2k$, then $\langle a_1,..., a_k\rangle$ is generated by any three consecutive elements in $a_1,..., a_k$, hence by Proposition 2.4, we may assume that for any subsequence T of length $m = 4, 5, 6$, or 7 of consecutive elements, we have $|\sum(T)| \geqslant 2m$. Since $k \geqslant 8$, $a_1,..., a_k$ may be partitioned into $r \geqslant 2$ consecutive blocks $T_1, T_2,..., T_r$, where the first $r - 1$ blocks have length 4, and the last has length 4, 5, 6, or 7. Then applying the Kemperman–Wehn theorem $r - 1$ times in $[\{0\} \cup \sum(T_1)] + [\{0\} \cup \sum(T_2)] + \cdots + [\{0\} \cup \sum(T_r)]$ gives $|\sum(S)| \geqslant 2k$, and this completes the proof.

References

1. R. B. Eggleton and P. Erdös, Two combinatorial problems in group theory, *Acta Arith.* **21** (1972), 111–116.
2. J. H. B. Kemperman, On complexes in a semigroup, *Indag. Math.* **18** (1956), 247–254.
3. J. E. Olson, Sums of sets of group elements, *Acta Arith.* **28** (1975), 147–156.
4. E. Szemerédi, On a conjecture of Erdös and Heilbronn, *Acta Arith.* **17** (1970), 227–229.
5. E. T. White, Ph.D. Dissertation, Pennsylvania State University, 1977.